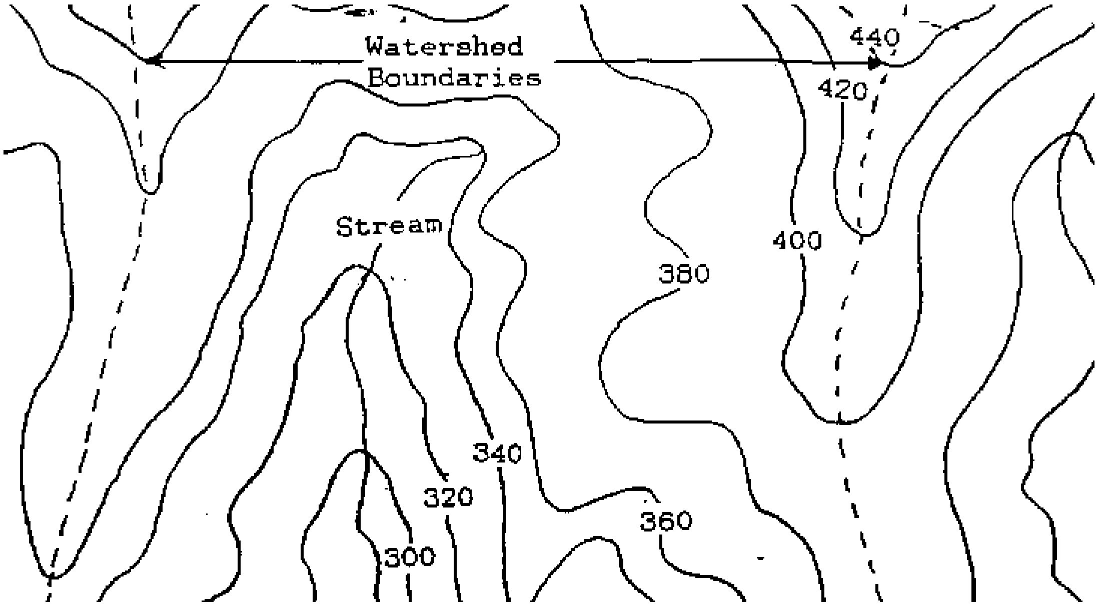
Tangency and Normals, 10/03/2013



This map shows a landscape using contour lines. The contour lines represent level sets of a function of two variables – the altitude at a particular longitude and latitude.

Running approximately perpendicular to the level curves are three other curves, marked as a stream and two watershed boundaries. These curves follow the path of steepest ascent (if we go from the bottom to the top). The tangent vectors to the stream and the watershed boundaries are given by the gradient of the altitude function. In general, the gradient of a function is perpendicular to its level curves and surfaces.

This example also shows some of the limitations of the gradient. From a point on the stream, the direction of steepest ascent is given by walking upstream. However, the easiest way to gain altitude fast is to go left or right, and climb the side of the valley. However, when walking down one side of the valley to the stream and up the other side, we see that the altitude function has a local minimum at the stream. In other words, the directional derivative to the left or the right is 0. Taking the derivative only gives us the instantaneous rate of change – it does not tell us that if we go in this direction we will soon begin to curve up. That needs information from the second derivative.

The equation $\nabla f(\mathbf{a}) \cdot (\mathbf{x} - \mathbf{a}) = 0$ gives the set of all vectors tangent to the level curve or surface of f passing through $\mathbf{a}$. The solution set is a tangent line or tangent plane. The gradient ∇f is the normal vector perpendicular to that line or plane.

Exercise: The curve $y^2 = 6 - 2x^2$ gives an ellipse. As a level curve, this is $f(x, y) = 2x^2 + y^2 = 6$. To find the tangent line at $(1, 2)$, we calculate $\nabla f = \langle 4x, 2y \rangle = \langle 4, 4 \rangle$ at $(1, 2)$. Then the equation is $\langle 4, 4 \rangle \cdot \langle x - 1, y - 2 \rangle = 4x + 4y - 12 = 0$. The line with direction $\langle 4, 4 \rangle$ is the "normal line" to the ellipse at $(1, 2)$, because it intersects the ellipse perpendicularly there.

Alternatively, we can calculate the slope of the line with implicit differentiation.

When considering tangents to a surface, there is an entire plane of tangent lines. For example, let us consider the point $(1, 1, 7)$ on the elliptic paraboloid $z = 2x^2 + 5y^2$. As a level surface, this is $f(x, y, z) = 2x^2 + 5y^2 - z = 0$. The gradient is $\nabla f = \langle 4x, 10y, -1\rangle$, and at $(1, 1, 7)$ this is $\langle 4, 10, -1\rangle$.

This vector must be perpendicular to the surface at $(1, 1, 7)$, so the line $\langle 1, 1, 7\rangle + t\,\langle 4, 10, -1\rangle$ is the normal line to the elliptic paraboloid at that point. The tangent plane is perpendicular to this line, so $\langle 4, 10, -1\rangle$ is the normal vector, giving $4x + 10y - z = 7$ for the tangent plane.

Exercise:

Consider the point $(2, 1, -1)$ on the surface $xy^3z^2 = 2$. This surface is already a level surface, $f(x, y, z) = xy^3z^2 = 2$. $\nabla f = \langle y^3z^2, 3xy^2z^2, 2xy^3z\rangle$, and at the point $(2, 1, -1)$ this is $\langle 1, 6, -4\rangle$. Therefore the tangent plane is $x + 6y - 4z = 12$.

When two surfaces intersect, the result is usually a curve. If a vector is tangent to that curve at the point P, it must be tangent to both surfaces at the point P as well. Therefore the tangent line will be the intersection of the two tangent planes. If one already knows a point on the line of intersection of two planes, the easiest way to calculate the direction vector is as the cross product of the two normals. So we have:

The tangent line to the curve of intersection of $f = 0$ and $g = 0$ has direction vector $\nabla f \times \nabla g$.

Consider the intersection of the cone $f(x, y, z) = x^2 + 3y^2 - 2z^2 = 0$ and the cylinder $g(x, y, z) = x^2+y^2-1 = 0$. To find the tangent line at $(0, 1, \sqrt{3/2})$ we find the gradients of the two functions at that point. $\nabla f = \langle 2x, 6y, -4z\rangle = \left\langle 0, 6, -4\sqrt{3/2}\right\rangle$ and $\nabla g = \langle 2x, 2y, 0\rangle = \langle 0, 2, 0\rangle$. The direction line is $\left\langle 0, 6, -4\sqrt{3/2}\right\rangle \times \langle 0, 2, 0\rangle = \left\langle 8\sqrt{3/2}, 0, 0\right\rangle$, or more simply $\langle 1, 0, 0\rangle$. The equation of the line is then $\left\langle 0, 1, \sqrt{3/2}\right\rangle + t\,\langle 1, 0, 0\rangle$.

Exercise: The plane $y + z = 3$ intersects the cylinder $f(x, y, z) = x^2 + y^2 = 5$ in an ellipse. To find the tangent line to this ellipse at the point $(1, 2, 1)$, we find normal vectors to the two tangent planes. For the first, we simply have $\langle 0, 1, 1\rangle$, because the plane is its own tangent plane. For the second, $\nabla f = \langle 2x, 2y, 0\rangle = \langle 2, 4, 0\rangle$ at the point. The cross product $\langle 0, 1, 1\rangle \times \langle 2, 4, 0\rangle = \langle -4, 2, -2\rangle$ and the line is $\langle 1, 2, 1\rangle + t\,\langle -4, 2, -2\rangle$.

Exercise: To find the normal line to the sphere $f(x, y, z) = x^2 + y^2 + z^2 = r^2$ through any point, it is enough to visualize the situation. If you stand on the surface of the earth, and travel perpendicular to the surface, then provided you travel downwards you will come to the center of the earth. So the normal line to the sphere at (x, y, z) is $t\,\langle x, y, z\rangle$. One can verify this by calculations: $\nabla f = \langle 2x, 2y, 2z\rangle$ so the line is $\langle x, y, z\rangle + t\,\langle 2x, 2y, 2z\rangle$, which can be rewritten as $t\,\langle x, y, z\rangle$.

Exercise: Are there any points on the hyperboloid $x^2 - y^2 - z^2 = 1$ where the tangent plane is parallel to the plane $z = x + y$? To check if two planes are parallel, we check whether their normal vectors are parallel, or scalar multiples of each other. The normal to the tangent plane is $\langle 2x, -2y, -2z \rangle$ and the normal to the given plane is $\langle 1, 1, -1 \rangle$. The planes are parallel if there is a scalar c such that $\langle 2x, -2y, -2z \rangle = c \langle 1, 1, -1 \rangle$. To find the point, we must solve the simultaneous equations for x, y, z, c:

$$\begin{aligned} x^2 - y^2 - z^2 &= 1 \\ 2x &= c \\ -2y &= c \\ -2z &= -c. \end{aligned}$$

We put x, y, z in terms of c: $x = c/2, y = -c/2, z = c/2$ and substitute into the first equation to get $c^2/4 - c^2/4 - c^2/4 = -c^2/4 = 1$. The left side is negative, the right side is positive, so there can be no solution. Therefore there is no point with a tangent plane parallel to the give one.

Exercise: To show that the ellipsoid $f(x, y, z) = 3x^2 + 2y^2 + z^2 = 9$ and the sphere $g(x, y, z) = x^2 + y^2 + z^2 - 8x - 6y - 8z + 24 = 0$ are tangent to each other at the point $(1, 1, 2)$ we must check they have the same tangent plane at $(1, 1, 2)$. We already know the tangent planes have the point $(1, 1, 2)$ in common, so we just need to check they have parallel normal vectors.

$\nabla f = \langle 6x, 4y, 2z \rangle = \langle 6, 4, 4 \rangle$ at the point, and $\nabla g = \langle 2x - 8, 2y - 6, 2z - 8 \rangle = \langle -6, -4, -4 \rangle$. Because $\nabla f = -\nabla g$, the vectors are parallel, the planes are the same, and the surfaces are indeed tangent to each other.